

THE MOST FREQUENT PEAK SET OF A RANDOM PERMUTATION

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ABSTRACT. Given a subset $S \subseteq \mathbb{P}$, let $P(S; n)$ be the number of permutations in the symmetric group of $\{1, 2, \dots, n\}$ that have peak set S . We prove a recent conjecture due to Billey, Burdzy and Sagan, which determines the sets that maximize $P(S; n)$, where S ranges over all subsets of $\{1, 2, \dots, n\}$.

1. INTRODUCTION

Given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ in \mathcal{S}_n , the symmetric group of the set $[n] = \{1, 2, \dots, n\}$, the up-down sequence of σ is the word $F(\sigma) = z_1 z_2 \cdots z_{n-1}$ with letters in $\{+, -\}^{n-1}$ defined by $z_i = -$ if i is a descent of σ (i.e., $\sigma_i > \sigma_{i+1}$) and $z_i = +$ if i is an ascent of σ (i.e., $\sigma_i < \sigma_{i+1}$). The enumeration of permutations with a given up-down sequence is a traditional topic of classical combinatorics which has received considerable interest since the end of the 19th century (see e.g. [8, 2, 3, 6, 9, 10]). A classical result in this topic that has been proved several times during the last century (see e.g. [2, 9, 10] but the list is not exhaustive) is that the maximum of $\beta(w)$, the number of permutations of $[n]$ with up-down sequence w , where w ranges over all up-down sequences of length $n-1$, occurs when w is an alternating sequence, i.e. $w = + - + - + - \cdots$ or $w = - + - + - + \cdots$. A refinement of this maximization problem which consists in maximizing $\beta(w)$ where w ranges over all up-down sequences of length n with a fixed number of alternating runs, has been conjectured by Gessel and was solved by Ehrenborg and Mahajan [4].

A related maximization problem has recently arisen in the work on the peak statistic by Billey, Burdzy and Sagan [1]. Recall that, given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ in \mathcal{S}_n , an index i is called a peak of σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. The peak set is defined to be

$$\mathcal{PS}(\sigma) = \{i : i \text{ is a peak of } \sigma\},$$

and given a set S of integers, we set

$$\mathcal{P}(S; n) = \{\sigma \in \mathcal{S}_n : \mathcal{PS}(\sigma) = S\},$$

and $P(S; n) = \#\mathcal{P}(S; n)$. For instance, if $\sigma = 265143$, we have $\mathcal{PS}(\sigma) = \{2, 5\}$ and thus $\sigma \in \mathcal{P}(\{2, 5\}; 6)$. The numbers $P(S; n)$ have been the center of the work of Billey, Burdzy and Sagan [1] and a conjecture on the sets S that maximize $P(S; n)$ among all subsets of $[n]$ has been proposed. The main purpose of the present paper is to solve this conjecture. To present their conjecture, it is more convenient to use the notion of peak-composition.

Suppose that the peak set of $\sigma \in \mathcal{S}_n$ is $\mathcal{PS}(\sigma) = \{i_1 < i_2 < \cdots < i_k\}$. Then, the composition $\text{pc}(\sigma) = (c_1, c_2, \dots, c_{k+1})$ of n defined by $c_j = i_j - i_{j-1}$ with $i_0 = 0$ and $i_{k+1} = n$ is called the peak-composition of σ . If \mathbf{c} is a composition of $[n]$, we set

$$\mathcal{P}(\mathbf{c}) = \{\sigma \in \mathcal{S}_n : \text{pc}(\sigma) = \mathbf{c}\},$$

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and $P(\mathbf{c}) = \#\mathcal{P}(\mathbf{c})$. For instance, if $\sigma = 265143$, we have $\text{pc}(\sigma) = \{2, 3, 1\}$ and thus $\sigma \in \mathcal{P}(2, 3, 1)$. Note that if $S = \{i_1 < i_2 < \dots < i_k\}$ is a subset of $[n]$ then $\mathcal{P}(S; n) = \mathcal{P}(\mathbf{c})$ where \mathbf{c} is defined as above. For instance, we have $\mathcal{P}(\{2, 5\}; 6) = \mathcal{P}(2, 3, 1)$. The size of a composition is the sum of its parts. In the sequel, we say that a composition \mathbf{c} is maximal if $P(\mathbf{b}) \leq P(\mathbf{c})$ for any composition \mathbf{b} having the same size as \mathbf{c} . The following result has been conjectured by Billey, Burdzy and Sagan.

Theorem 1.1. *For $n \geq 1$, let $\mathcal{C}^*(n)$ denote the set of maximal compositions of n . For $\ell \geq 2$, we have*

- $\mathcal{C}^*(3\ell) = \{(3^\ell), (4, 3^{\ell-2}, 2)\},$
- $\mathcal{C}^*(3\ell + 1) = \{(3^s, 2, 3^t, 2) : s \geq 1, t \geq 0, s + t = \ell - 1\},$
- $\mathcal{C}^*(3\ell + 2) = \{(3^\ell, 2)\}.$

Equivalently, for $n \geq 6$, the sets that maximize $P(S; n)$ are

- if $n \equiv 0 \pmod{3}$, $\{3, 6, 9, \dots\} \cap \{1, 2, \dots, n-1\}$ and $\{4, 7, 10, \dots\} \cap \{1, 2, \dots, n-1\},$
- if $n \equiv 1 \pmod{3}$, $\{3, 6, 9, \dots, 3s, 3s+2, 3s+5, \dots\} \cap \{1, 2, \dots, n-1\}$ with $1 \leq s \leq \lfloor \frac{n}{3} \rfloor - 1,$
- if $n \equiv 2 \pmod{3}$, $\{3, 6, 9, \dots\} \cap \{1, 2, \dots, n-1\}.$

In the above theorem, exponent just indicates iteration. For instance, $(3^2, 4)$ is for $(3, 3, 4)$. We will also determine the cardinality of $\mathcal{P}(\mathbf{c})$ when \mathbf{c} is a maximal composition. The expressions obtained are extremely simple.

Theorem 1.2. *Suppose $n \geq 6$ and $\mathbf{c} \in \mathcal{C}^*(n)$. Set $\ell = \lfloor \frac{n}{3} \rfloor$. Then, we have,*

- if $n \equiv 0 \pmod{3}$, $P(\mathbf{c}) = \frac{1}{5}3^{2-\ell}n!,$
- if $n \equiv 1 \pmod{3}$, $P(\mathbf{c}) = \frac{2}{5}3^{1-\ell}n!,$
- if $n \equiv 2 \pmod{3}$, $P(\mathbf{c}) = 3^{-\ell}n!.$

Our proof of Theorem 1.1 is essentially based on some comparison lemmas presented in Section 2 and a counting formula for the number $P(\mathbf{c})$ which is efficient when \mathbf{c} contains only short patterns of elements distinct from 3. In Section 3, we show that a maximal composition can not have a part greater than 4. In Section 4, we provide further patterns that a maximal composition must avoid. In Section 5, we preset our counting formula from which we deduce Theorem 1.2. Finally, the proof of Theorem 1.1 is completed in Section 6.

2. COMPARISON LEMMAS

In this section, we provide some comparison lemmas. We first set up some additional terminology. The concatenation of two compositions \mathbf{c}_1 and \mathbf{c}_2 is denoted by $\mathbf{c}_1 \oplus \mathbf{c}_2$. For instance, $(4, 3, 2) \oplus (2, 3) = (4, 3, 2, 2, 3)$. We assume the existence of an empty (with 0 part) composition ϵ such that $\mathbf{c} \oplus \epsilon = \epsilon \oplus \mathbf{c} = \mathbf{c}$. We say that a composition \mathbf{c} is admissible if $\mathcal{P}(\mathbf{c}) \neq \emptyset$. It is not hard to prove the following result.

Fact 2.1. *A composition $\mathbf{c} = (c_1, \dots, c_k)$ is admissible if and only if $k = 1$ or $c_1, \dots, c_{k-1} \geq 2$ and $c_k \geq 1$.*

Given a composition $\mathbf{c} = (c_1, c_2, \dots, c_k)$, we set $r'\mathbf{c} = \mathbf{c}$ if $k = 1$ and $r'\mathbf{c} = (c_k + 1, c_{k-1}, \dots, c_2, c_1 - 1)$ if $k \geq 2$. Considering the reverse of a permutation, we obtain the following result.

Fact 2.2. *For any composition \mathbf{c} , we have $P(\mathbf{c}) = P(r'\mathbf{c})$. In particular, \mathbf{c} is maximal if and only if $r'\mathbf{c}$ is maximal.*

For $a, b \in \mathbb{P}$ and \mathbf{c} a composition of an integer $n \geq 1$, we let

$$\begin{aligned} Ini_{a,b}(\mathbf{c}) &= \{\sigma \in \mathcal{P}(\mathbf{c}) : a = \sigma_1, \sigma_{n-1} < \sigma_n = b\}, \\ Int_{a,b}(\mathbf{c}) &= \{\sigma \in \mathcal{P}(\mathbf{c}) : a = \sigma_1 > \sigma_2, \sigma_{n-1} < \sigma_n = b\}, \end{aligned}$$

and $Ini_{\cdot,b}(\mathbf{c}) = \bigcup_{a=1}^n Ini_{a,b}(\mathbf{c})$. We set $Ini_{a,b} = \#Ini_{a,b}$, $Ini_{\cdot,b} = \#Ini_{\cdot,b}$ and $Int_{a,b} = \#Int_{a,b}$. We also let $1 + \mathbf{c}$ denote the composition of $n + 1$ obtained from \mathbf{c} by increasing the first part of \mathbf{c} by 1. So, if $\mathbf{c} = (c_1, \dots, c_k)$, $1 + \mathbf{c} = (c_1 + 1, c_2, \dots, c_k)$. For instance, $1 + (2, 4, 3, 2) = (3, 4, 3, 2)$.

If $\mathbf{x} = (x_1, x_2, \dots, x_k)$ and $\mathbf{y} = (y_1, y_2, \dots, y_k)$ are two sequences of the same length, we write $\mathbf{x} < \mathbf{y}$ if $x_i \leq y_i$ for $1 \leq i \leq k$ and $\mathbf{x} \neq \mathbf{y}$.

Proposition 2.3. *Let \mathbf{c} be a composition of an integer $n \geq 1$. Suppose that there exists a composition \mathbf{c}' of n such that*

(1) $Int_{a,b}(1 + \mathbf{c}) \leq Int_{a,b}(1 + \mathbf{c}')$ for all $a, b \in [n + 1]$ and there are two indices u, v such that $Int_{u,v}(1 + \mathbf{c}) < Int_{u,v}(1 + \mathbf{c}')$: then, for any nonempty compositions \mathbf{a} and \mathbf{b} such that $\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b}$ is admissible,

$$P(\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b}) < P(\mathbf{a} \oplus \mathbf{c}' \oplus \mathbf{b}),$$

and thus, $\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b}$ is not maximal.

(2) $(Ini_{\cdot,b}(\mathbf{c}))_{1 \leq b \leq n} < (Ini_{\cdot,b}(\mathbf{c}'))_{1 \leq b \leq n}$: then, for any nonempty composition \mathbf{b} such that $\mathbf{c} \oplus \mathbf{b}$ is admissible,

$$P(\mathbf{c} \oplus \mathbf{b}) < P(\mathbf{c}' \oplus \mathbf{b}),$$

and thus, $\mathbf{c} \oplus \mathbf{b}$ is not maximal.

To simplify the readability of the proof of the above result, we first state a preliminary lemma. In the sequel, a permutation $\tau = \tau_1 \cdots \tau_n$ of a set $S \subset \mathbb{P}$ with cardinality n is said to be order-isomorphic to a permutation $\sigma = \sigma_1 \cdots \sigma_n$ in \mathcal{S}_n if for $1 \leq i < j \leq n$, $\tau_i < \tau_j$ is equivalent to $\sigma_i < \sigma_j$.

Lemma 2.4. *For $i = 1, 2, 3$, let $\mathbf{c}^{(i)}$ be a composition of a positive integer n_i such that $\mathbf{c}^{(1)} \oplus \mathbf{c}^{(2)} \oplus \mathbf{c}^{(3)}$ is admissible. Suppose $Int_{u,v}(1 + \mathbf{c}^{(2)}) > 0$, then for each permutation $\tau \in Int_{u,v}(1 + \mathbf{c}^{(2)})$ there exists a permutation σ in $\mathcal{P}(\mathbf{c}^{(1)} \oplus \mathbf{c}^{(2)} \oplus \mathbf{c}^{(3)})$ such that $\sigma_{n_1} \sigma_{n_1+1} \cdots \sigma_{n_1+n_2}$ is order isomorphic to τ .*

Proof. We first choose two permutations $\gamma \in Ini_{\cdot,n_1}(\mathbf{c}^{(1)})$ and $\beta \in Ini_{\cdot,n_3+1}(r'(1 + \mathbf{c}^{(3)}))$ (this choice is always possible since, as it is easily seen, $Int_{\cdot,n}(\mathbf{c}) \neq \emptyset$ for any composition \mathbf{c} of n). Then set $\sigma_1, \dots, \sigma_{n_1-1} = \gamma_1, \dots, \gamma_{n_1-1}$ (resp., $\sigma_{n_1+n_2+n_3}, \sigma_{n_1+n_2+n_3-1}, \dots, \sigma_{n_1+n_2+1} = \beta_1+n_1-1, \dots, \beta_{n_3}+n_1-1$). We then set $\sigma_{n_1}, \sigma_{n_1+1}, \dots, \sigma_{n_1+n_2} = \tau_1+n_1+n_3-1, \tau_2+n_1+n_3-1, \dots, \tau_{n_2+1}+n_1+n_3-1$. It is easily checked that the permutation $\sigma = \sigma_1 \cdots \sigma_{n_1+n_2+n_3}$ is in $\mathcal{P}(\mathbf{c}^{(1)} \oplus \mathbf{c}^{(2)} \oplus \mathbf{c}^{(3)})$ and $\sigma_{n_1} \sigma_{n_1+1} \cdots \sigma_{n_1+n_2}$ is order isomorphic to τ . \square

Proof of Proposition 2.3. (1) Let \mathbf{a} and \mathbf{b} be two compositions of positive integers r and s such that $\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b}$ is admissible. By our hypothesis, we can consider a family of injections $(\phi_{c,d})_{1 \leq c, d \leq n+1}$, $\phi_{c,d} : Int_{c,d}(1 + \mathbf{c}) \mapsto Int_{c,d}(1 + \mathbf{c}')$, such that $\phi_{u,v}$ is not surjective. Then, consider the function Γ which associates to a permutation $\sigma \in \mathcal{P}(\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b})$ the permutation σ' defined by $\sigma'_1, \dots, \sigma'_{r-1} = \sigma_1, \dots, \sigma_{r-1}$, $\sigma'_{r+n+1}, \dots, \sigma'_{r+n+s} = \sigma_{r+n+1}, \dots, \sigma_{r+n+s}$, and $\sigma'_r \sigma'_{r+1} \cdots \sigma'_{r+n}$ is the unique permutation of the set $\{\sigma_r, \sigma_{r+1}, \dots, \sigma_{r+n}\}$ which is order isomorphic to $\phi_{\tau_1, \tau_{n+1}}(\tau)$, where τ is the permutation of $[n + 1]$ which is order isomorphic to $\sigma_r \sigma_{r+1} \cdots \sigma_{r+n}$. It is easily checked that Γ is a well-defined function from $\mathcal{P}(\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b})$ to $\mathcal{P}(\mathbf{a} \oplus \mathbf{c}' \oplus \mathbf{b})$ which is injective but, by Lemma 2.4, not surjective.

The proof of (2) is easier than and very similar to the proof of (1), and so is left to the reader. \square

A simple but important consequence of the above result is given in the next section.

3. THE LARGEST PART IN A MAXIMAL COMPOSITION

Proposition 3.1. *Let $\mathbf{c} = (c_1, \dots, c_k)$ be a composition of n . If $k = 1$ and $n \geq 5$, then \mathbf{c} is not maximal. If $k \geq 2$ and there exists i , $1 \leq i \leq k-1$, such that $c_i \geq 5$ or $c_k \geq 4$, then \mathbf{c} is not maximal.*

The above result is an immediate consequence of Proposition 2.3 and the two following lemmas.

Lemma 3.2. *For $n \geq 3$, we have*

- (1a) $\text{Int}_{n,a}(n) = \text{Int}_{a,n}(n) = 2^{a-2}$ if $2 \leq a \leq n-1$,
- (1b) $\text{Int}_{a,b}(n) = 0$ if $a \neq n$ and $b \neq n$, or $a = 1$, or $b = 1$;
- (2a) $\text{Ini}_{1,n}(n) = 1$ and $\text{Ini}_{1,b}(n) = 0$ if $b < n$,
- (2b) $\text{Ini}_{a,b}(n) = \text{Int}_{a,b}(n)$ if $a \neq 1$.

Proof. (1) Suppose $2 \leq a \leq n-1$. A permutation σ in \mathcal{S}_n is in $\text{Int}_{n,a}(n)$ if and only if it writes $\sigma = n w_1 1 w_2 a$ where w_1 is decreasing and w_2 is increasing. There are 2^{a-2} choices for w_1 and each choice of w_1 determines uniquely w_2 . The second assertion is left to the reader.

(2a) Clearly, for $n \geq 2$, $\text{Ini}_{1,n}(n) = \{1 2 \cdots n\}$. If $b < n$ and $\sigma \in \mathcal{S}_n$ with $\sigma_1 = 1$ and $\sigma_n = b$, then σ has at least one peak (the index i such that $\sigma_i = n$) and thus $\text{pc}(\sigma) \neq (n)$, whence $\text{Ini}_{1,b}(n) = \emptyset$.

(2b) By definition, for all integers a, b , we have $\text{Int}_{a,b}(n) \subseteq \text{Ini}_{a,b}(n)$. If $b = 1$, $\text{Ini}_{a,b}(n) = \emptyset$. Suppose $a \neq 1$, $b \neq 1$ and $\sigma \in \text{Ini}_{a,b}(n)$. If $\sigma \notin \text{Int}_{a,b}(n)$ (i.e., if $\sigma_1 < \sigma_2$), the greatest integer p such that $\sigma_1 < \sigma_2 < \cdots < \sigma_p$ is less than $n-1$ since $\sigma_i = 1$ for some index i such that $2 \leq i \leq n-1$. This implies that p is a peak of σ and thus $\text{pc}(\sigma) \neq (n)$. Consequently, $\text{Ini}_{a,b}(n) \subseteq \text{Int}_{a,b}(n)$. \square

Lemma 3.3. *For $n \geq 5$, we have*

- (1a) $\text{Int}_{n,n-1}(3, n-3) = \text{Int}_{n-1,n}(3, n-3) = (n-4)2^{n-4}$,
- (1b) $\text{Int}_{n,a}(3, n-3) = (n-2-a)2^{a-2} + \chi(a \geq 3)(a-1)2^{a-3}$ for $2 \leq a \leq n-2$,
- (1c) $\text{Int}_{a,n}(3, n-3) = \text{Int}_{a,n-1}(3, n-3) + (a-1)2^{n-5}$ for $2 \leq a \leq n-2$;
- (2) $\text{Ini}_{1,n}(3, n-3) > 1$.

Proof. (1a) Clearly, $\text{Int}_{n,n-1}(3, n-3) = \text{Int}_{n-1,n}(3, n-3)$. A permutation σ in \mathcal{S}_n is in $\text{Int}_{n,n-1}(3, n-3)$ if and only if it writes $\sigma = n x k w_1 \ell w_2 n-1$ where w_1 is decreasing and w_2 is increasing, $1 \leq x < k$, $1 \leq \ell \leq k \leq n-2$ and all the letters of w_1 and w_2 are greater than ℓ . It is easily checked that for each k , there are $(k-1)2^{k-3}$ such permutations. Thus, $\text{Int}_{n,n-1}(3, n-3) = \sum_{k=3}^{n-2} (k-1)2^{k-3} = (n-4)2^{n-4}$.

(1b) Suppose $2 \leq a \leq n-2$. Clearly, if $\sigma \in \text{Int}_{n,a}(3, n-3)$, then we have $\sigma_3 = n-1$. We classify the permutations σ in $\text{Int}_{n,n-1}(3, n-3)$ according to the value of σ_2 . If $a < k \leq n-2$, then there are 2^{a-2} choices for the permutations σ in $\text{Int}_{n,a}(3, n-3)$ with $\sigma_2 = k$. If $1 \leq k < a$ and $a \geq 3$ (resp., $a = 2$), then there are 2^{a-3} (resp., 0) choices.

(1c) Suppose $2 \leq a \leq n-2$. We classify the permutations σ in $\text{Int}_{a,n}(3, n-3)$ according to the value of σ_{n-1} . If $\sigma_{n-1} = n-1$, then there are $\text{Int}_{a,n-1}(3, n-3)$ such permutations. If $\sigma_{n-1} \neq n-1$, then $\sigma_3 = n-1$ and there are $a-1$ choices for σ_2 . For each choice of σ_2 , there are 2^{n-5} corresponding permutations.

(2) It suffices to observe that $\{1 2 4 3 5 6 \cdots n, 1 3 4 2 5 6 \cdots n\} \subseteq \text{Ini}_{1,n}(3, n-3)$. \square

Proof of Proposition 3.1. (1) Suppose first $k \geq 2$. It follows from Lemmas 3.2 and 3.3 that

- for $n \geq 6$ and all $a, b \in [n]$, we have $\text{Int}_{a,b}(n) \leq \text{Int}_{a,b}(3, n-3)$, and $\text{Int}_{2,n}(n) < \text{Int}_{2,n}(3, n-3)$,
- for $n \geq 5$ and all $b \in [n]$, $\text{Ini}_{\cdot,b}(n) \leq \text{Ini}_{\cdot,b}(3, n-3)$, and $\text{Ini}_{\cdot,n}(n) < \text{Ini}_{\cdot,n}(3, n-3)$.

Consequently, by Proposition 2.3, if $n \geq 5$, then for any nonempty compositions \mathbf{a} and \mathbf{b} , the compositions $\mathbf{a} \oplus (n) \oplus \mathbf{b}$ and $(n) \oplus \mathbf{b}$ are not maximal. In other words, if \mathbf{c} is maximal then $c_i \leq 4$ for $i = 1, \dots, k-1$. This, combined with Fact 2.2, implies that if \mathbf{c} is maximal, $c_k \leq 3$.

(2) If $k = 1$, then a simple counting (see the proof of the above lemmas or [1]) shows that $P(n) = 2^{n-1}$ and $P(3, n-3) = (\binom{n-1}{2} - 1)2^{n-2}$, from which we deduce that $P(n) < P(3, n-3)$ for $n \geq 5$. \square

4. SOME FORBIDDEN PATTERNS IN A MAXIMAL COMPOSITION

The 3-factorization of a composition \mathbf{c} is the (unique) factorization of \mathbf{c} as

$$\mathbf{c} = \mathbf{c}^{(0)} \oplus (3) \oplus \mathbf{c}^{(1)} \oplus \dots \oplus (3) \oplus \mathbf{c}^{(k)}$$

where k is the number of parts in \mathbf{c} equal to 3 and the compositions $\mathbf{c}^{(i)}$ (possibly empty), called factors, have no part equal to 3. For instance, $(4, 4, 3, 2, 4, 2, 3, 3, 2, 1) = (4, 4) \oplus (3) \oplus (2, 4, 2) \oplus (3) \oplus (3) \oplus (2, 1)$.

The purpose of this section is to prove the following result.

Proposition 4.1. *Suppose \mathbf{c} is a maximal composition with at least 3 parts. Then,*

$$(4.1) \quad \mathbf{c} = x_0 \oplus (3) \oplus x_1 \oplus \dots \oplus (3) \oplus x_k$$

for some $k \geq 1$ and some sequence of compositions $(x_i)_i$ such that

$$x_0 \in \{\epsilon, (4)\}, x_k \in \{\epsilon\} \cup \{(2^s) : s \geq 1\}, \text{ and, for } i = 1, \dots, k-1, x_i \in \{\epsilon\} \cup \{(2^s), (4^t) : s, t \geq 1\}.$$

Clearly, in view of Proposition 3.1 and Fact 2.1, the above result is immediate from the following lemma.

Lemma 4.2. *For any nonempty compositions \mathbf{a} and \mathbf{b} ,*

- (1) *if $\mathbf{c} \in \{(2, 2), (2, 3), (2, 4)\}$, the composition $\mathbf{c} \oplus \mathbf{b}$ is not maximal;*
- (2) *if $\mathbf{c} \in \{(2, 1), (3, 1), (4, 1)\}$, the composition $\mathbf{a} \oplus \mathbf{c}$ is not maximal;*
- (3) *if $\mathbf{c} \in \{(2, 4), (4, 2)\}$, the compositions $\mathbf{c} \oplus \mathbf{b}$, $\mathbf{a} \oplus \mathbf{c}$ and $\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b}$ are not maximal;*
- (4) *the composition $(4, 4) \oplus \mathbf{b}$ is not maximal.*

Proof. For simplicity, if \mathbf{c} is a composition of n , the sequence $(\text{Ini}_{\cdot,b}(\mathbf{c}))_{1 \leq b \leq n}$ will be denoted by $T \mathbf{c}$. Using a computer algebra system, it is easy to obtain

$$\begin{aligned} T(2, 2) &= [0, 1, 2, 2], & T(4) &= [0, 1, 2, 4], & T(2, 3) &= [0, 2, 4, 6, 8], & T(3, 2) &= [0, 3, 6, 8, 8], \\ T(2, 4) &= [0, 3, 6, 10, 16, 24], & T(4, 2) &= [0, 7, 14, 20, 24, 24], & T(3, 3) &= [0, 8, 16, 24, 32, 40], \\ T(3, 4) &= [0, 15, 30, 48, 72, 104, 144], & T(4, 3) &= [0, 24, 48, 72, 96, 120, 144], \\ T(4, 4) &= [0, 55, 110, 172, 248, 344, 464, 608], & T(3, 2, 3) &= [0, 96, 192, 288, 384, 480, 576, 672]. \end{aligned}$$

from which it results that

$$T(2, 2) < T(4), \quad T(2, 3) < T(3, 2), \quad T(2, 4) < T(4, 2) < T(3, 3), \quad T(3, 4) < T(4, 3), \quad T(4, 4) < T(3, 2, 3).$$

By Proposition 2.3(2), these inequalities imply that if $\mathbf{c} \in \{(2, 2), (2, 3), (2, 4), (4, 2), (3, 4), (4, 4)\}$, the composition $\mathbf{c} \oplus \mathbf{b}$ is not maximal for any $\mathbf{b} \neq \epsilon$. By Fact 2.2, this can be reformulated as follows: if $\mathbf{c} \in \{(2, 1), (3, 1), (4, 1), (2, 3), (4, 2), (4, 3)\}$, the composition $\mathbf{a} \oplus \mathbf{c}$ is not maximal for any $\mathbf{a} \neq \epsilon$.

$a \setminus b$	2	3	4	5	6	7
2	[0, 0]	[0, 0]	[1, 2]	[2, 4]	[3, 6]	[3, 8]
3	[0, 0]	[0, 0]	[2, 4]	[4, 8]	[6, 12]	[6, 16]
4	[1, 2]	[2, 4]	[0, 0]	[6, 12]	[10, 18]	[10, 24]
5	[2, 4]	[4, 8]	[6, 12]	[0, 0]	[16, 24]	[16, 32]
6	[4, 6]	[8, 12]	[12, 18]	[16, 24]	[0, 0]	[24, 40]
7	[7, 8]	[14, 16]	[20, 24]	[24, 32]	[24, 40]	[0, 0]

TABLE 1. The values $[\text{Int}_{a,b}(5, 2), \text{Int}_{a,b}(4, 3)]$

Thus, to conclude our proof, it just remains to check that if $\mathbf{c} \in \{(2, 4), (4, 2)\}$, the composition $\mathbf{a} \oplus \mathbf{c} \oplus \mathbf{b}$ is not maximal for any $\mathbf{a} \neq \epsilon$ and any $\mathbf{b} \neq \epsilon$. Again, by using a computer algebra system, we can explicitly compute the array $([\text{Int}_{a,b}(5, 2), \text{Int}_{a,b}(4, 3)])_{1 \leq a, b \leq 7}$ (see Table 1) from which it results that for all integers a, b we have $\text{Int}_{a,b}(5, 2) \leq \text{Int}_{a,b}(4, 3)$ and (at least) one of these inequalities is strict. By Proposition 2.3(1), this implies that $\mathbf{a} \oplus (4, 2) \oplus \mathbf{b}$ is not maximal for any $\mathbf{a} \neq \epsilon$ and any $\mathbf{b} \neq \epsilon$. By Fact 2.2, the same is true with $(4, 2)$ replaced by $(2, 4)$. \square

Before we end this section, we provide further patterns that a maximal composition must avoid.

Lemma 4.3. *For any nonempty compositions \mathbf{a} and \mathbf{b} ,*

- (1) *if $\mathbf{c} \in \{(4, 3, 2), (4, 3, 4), (3, 2, 3, 2), (3, 3, 2, 3, 2)\}$, the composition $\mathbf{c} \oplus \mathbf{b}$ is not maximal;*
- (2) *if $\mathbf{c} \in \{(2, 3, 3), (4, 3, 3), (2, 3, 2, 2), (2, 3, 2, 3, 2)\}$, the composition $\mathbf{a} \oplus \mathbf{c}$ is not maximal.*

Proof. By Fact 2.2, the two assertions are equivalent. So it suffices to prove (1). Using a computer algebra system, we obtain

$$\begin{aligned}
T(4, 3, 2) &= [0, 504, 1008, 1488, 1920, 2280, 2544, 2688, 2688], \\
T(3, 3, 3) &= [0, 560, 1120, 1680, 2240, 2800, 3360, 3920, 4480], \\
T(4, 3, 4) &= [0, 9072, 18144, 27720, 38304, 50376, 64368, 80640, 99456, 120960, 145152], \\
T(3, 2, 3, 3) &= [0, 16128, 32256, 48384, 64512, 80640, 96768, 112896, 129024, 145152, 161280], \\
T(3, 2, 3, 2) &= [0, 2688, 5376, 7968, 10368, 12480, 14208, 15456, 16128, 16128], \\
T(4, 3, 3) &= [0, 2688, 5376, 8064, 10752, 13440, 16128, 18816, 21504, 24192], \\
T(3, 3, 2, 3, 2) &= [0, 887040, 1774080, 2644992, 3483648, 4273920, 4999680, 5644800, 6193152, \\
&\quad 6628608, 6935040, 7096320, 7096320], \\
T(4, 3, 3, 3) &= [0, 887040, 1774080, 2661120, 3548160, 4435200, 5322240, 6209280, 7096320, \\
&\quad 7983360, 8870400, 9757440, 10644480],
\end{aligned}$$

from which it results that

$$T(4, 3, 2) < T(3, 3, 3), \quad T(4, 3, 4) < T(3, 2, 3, 3), \quad T(3, 2, 3, 2) < T(4, 3, 3), \quad T(3, 3, 2, 3, 2) < T(4, 3, 3, 3).$$

This, by Proposition 2.3(2), gives the first assertion. \square

We can even go further with the same method but it will be more convenient and even more simpler to use the tools presented in the next section.

5. A COUNTING LEMMA

The purpose of this section is to provide an efficient counting formula for the number $P(\mathbf{c})$ when \mathbf{c} is a composition in which the factors of its 3-factorization have a small size. In the sequel, we let $|\mathbf{c}|$ denote the size of the composition \mathbf{c} . The size of the empty composition is given by $|\epsilon| = 0$. The following is the key result in this section.

Lemma 5.1. *Let \mathbf{a} and \mathbf{b} two compositions. If $\mathbf{b} \neq \epsilon$, then we have*

$$(5.1) \quad P(\mathbf{a} \oplus (3) \oplus \mathbf{b}) = \binom{|\mathbf{a}| + |\mathbf{b}| + 3}{|\mathbf{a}| + 1} P(\mathbf{a} \oplus (1)) P((2) \oplus \mathbf{b}).$$

Proof. Given a composition \mathbf{c} of a positive integer n and a finite subset $F \subset \mathbb{P}$ with cardinality $\#F = n$, we let $\mathcal{P}(\mathbf{c}; F)$ be the set of permutations of the set F having peak set equal to F . With this terminology, we have $\mathcal{P}(\mathbf{c}) = \mathcal{P}(\mathbf{c}; [n])$.

Let r and s be the sizes of the compositions \mathbf{a} and \mathbf{b} , and let

$$R_{\mathbf{a}, \mathbf{b}} = \{(\gamma, \beta) : \gamma \in \mathcal{P}(\mathbf{a} \oplus (1); S), \beta \in \mathcal{P}((2) \oplus \mathbf{b}; T), S \cup T = \{1, \dots, r + s + 3\}\}.$$

Note that $\#R_{\mathbf{a}, \mathbf{b}}$ is equal to the right-hand side of (5.1). So, to prove (5.1), it suffices to present a bijection between $R_{\mathbf{a}, \mathbf{b}}$ and $\mathcal{P}(\mathbf{a} \oplus (3) \oplus \mathbf{b})$. This is quite easy.

Consider the concatenation function $\Gamma : R_{\mathbf{a}, \mathbf{b}} \mapsto \mathcal{S}_{r+s+3}$ defined by $\Gamma(\gamma, \beta) = \gamma\beta$. It is clear that Γ is well defined, injective and $\Gamma(R_{\mathbf{a}, \mathbf{b}}) \supseteq \mathcal{P}(\mathbf{a} \oplus (3) \oplus \mathbf{b})$. We now check that $\Gamma(R_{\mathbf{a}, \mathbf{b}}) \subseteq \mathcal{P}(\mathbf{a} \oplus (3) \oplus \mathbf{b})$. Suppose $\sigma = \gamma\beta$ with $(\gamma, \beta) \in R_{\mathbf{a}, \mathbf{b}}$. To prove that $\sigma = \gamma\beta$ is in $\mathcal{P}(\mathbf{a} \oplus (3) \oplus \mathbf{b})$, it suffices to prove that $r + 1$ and $r + 2$ are not peaks of σ which is immediate since $\sigma_r = \gamma_r > \gamma_{r+1} = \sigma_{r+1}$ (r is a peak of γ) and $\sigma_{r+2} = \beta_1 < \beta_2 = \sigma_{r+3}$ (2 is a peak of β). \square

Repeated applications of Lemma 5.1 leads to the following result the proof of which is omitted.

Proposition 5.2. *Let $(\mathbf{c}^{(i)})_{0 \leq i \leq k}$ be a sequence of compositions such that $\mathbf{c}^{(k)} \neq \epsilon$. Then, we have*

$$(5.2) \quad P(\mathbf{c}^{(0)} \oplus (3) \oplus \mathbf{c}^{(1)} \oplus (3) \cdots \oplus \mathbf{c}^{(k)}) = \binom{\sum_{i=0}^k |\mathbf{c}^{(i)}| + 3k}{|\mathbf{c}^{(0)}| + 1, 3 + |\mathbf{c}^{(1)}|, \dots, 3 + |\mathbf{c}^{(k-1)}|, |\mathbf{c}^{(k)}| + 2} \\ \times P(\mathbf{c}^{(0)} \oplus (1)) \left(\prod_{i=1}^{k-1} P((2) \oplus \mathbf{c}^{(i)} \oplus (1)) \right) P((2) \oplus \mathbf{c}^{(k)}).$$

In the sequel, we will use the following formulas:

$$(5.3) \quad \begin{aligned} P(2, 1) &= 2, & P(3, 1) &= 8, & P(4, 1) &= 24, & P(2, 2, 1) &= 16, \\ P(2, 2) &= 8, & P(2, 3) &= 24, & P(2, 2, 2) &= 96, & P(2, 4) &= 64, \\ P(2, 2, 1) &= 16, & P(2, 2, 2, 1) &= 272, & P(2, 4, 1) &= 288. \end{aligned}$$

The following non maximality criterion is immediate from the above result.

Proposition 5.3. *Suppose $\mathbf{c} = \mathbf{c}^{(0)} \oplus (3) \oplus \mathbf{c}^{(1)} \oplus (3) \cdots \oplus \mathbf{c}^{(k)}$ with $k \geq 1$ and $\mathbf{c}^{(k)} \neq \epsilon$. If there exists a composition \mathbf{b} such that*

- (1) $|\mathbf{b}| = |\mathbf{c}^{(i)}|$ for some index i , $1 \leq i \leq k - 1$, and $P((2) \oplus \mathbf{c}^{(i)} \oplus (1)) < P((2) \oplus \mathbf{b} \oplus (1))$, then \mathbf{c} is not maximal;
- (2) $|\mathbf{b}| = |\mathbf{c}^{(k)}|$ and $P((2) \oplus \mathbf{c}^{(k)}) < P((2) \oplus \mathbf{b})$, then \mathbf{c} is not maximal.

Corollary 5.4. *Suppose \mathbf{c} is a maximal composition with at least 3 parts and with 3-factorization*

$$(5.4) \quad \mathbf{c} = x_0 \oplus (3) \oplus x_1 \oplus \cdots \oplus (3) \oplus x_k.$$

Then, $k \geq 1$ and the sequence of compositions $(x_i)_i$ satisfies

$$x_0 \in \{\epsilon, (4)\}, x_k \in \{\epsilon, (2), (2, 2)\}, \text{ and } x_i \in \{\epsilon, (2), (4)\} \text{ for } i = 1, \dots, k-1.$$

Proof. By Proposition 4.1, we can assume that $\mathbf{c} = x_0 \oplus (3) \oplus x_1 \oplus \cdots \oplus (3) \oplus x_k$ with $k \geq 1$ and $x_0 \in \{\epsilon, (4)\}$, $x_k \in \{\epsilon\} \cup \{(2^s) : s \geq 1\}$, and $x_i \in \{\epsilon\} \cup \{(2^s), (4^t) : s, t \geq 1\}$ for $i = 1, \dots, k-1$.

Using a computer algebra system, we obtain

$$T(2, 2, 2, 2) = [0, 61, 122, 178, 224, 256, 272, 272], \quad T(2, 3, 3) = [0, 80, 160, 240, 320, 400, 480, 560]$$

$$T(2, 4, 4) = [0, 889, 1778, 2726, 3792, 5032, 6496, 8224, 10240, 12544],$$

$$T(2, 3, 2, 3) = [0, 1792, 3584, 5376, 7168, 8960, 10752, 12544, 14336, 16128],$$

from which it results that $T((2) \oplus (2, 2, 2)) < T((2) \oplus (3, 3))$ and $T((2) \oplus (4, 4)) < T((2) \oplus (3, 2, 3))$. Moreover, it is easily checked that $P((2) \oplus (2, 2, 2)) < P((2) \oplus (3, 3))$. These inequalities combined with Propositions 2.3(2) and 5.3 imply that $x_k \in \{\epsilon, (2), (2, 2)\}$, and $x_i \in \{\epsilon, (2), (2, 2), (4)\}$ for $i = 1, \dots, k-1$.

To complete our proof, it remains to prove that if $\mathbf{c} = \mathbf{a} \oplus (3) \oplus (2, 2) \oplus (3) \oplus \mathbf{b}$, then \mathbf{c} is not maximal. Suppose $\mathbf{b} = \epsilon$, then $r'\mathbf{c} = (4, 2) \oplus \mathbf{u}$ for some composition $\mathbf{u} \neq \epsilon$. Then, by Lemma 4.2 and Fact 2.2, $r'\mathbf{c}$ and \mathbf{c} are not maximal. If $\mathbf{b} \neq \epsilon$, it results from (5.2) and (5.3) that

$$(5.5) \quad \frac{P(\mathbf{a} \oplus (3) \oplus (2, 2) \oplus (3) \oplus \mathbf{b})}{P(\mathbf{a} \oplus (3) \oplus (4) \oplus (3) \oplus \mathbf{b})} = \frac{P((2) \oplus (2, 2) \oplus (1))}{P((2) \oplus (4) \oplus (1))} = \frac{272}{288} < 1.$$

This concludes the proof. \square

Before we end this section, we prove Theorem 1.2. It is convenient to present a slightly more general result than Proposition 5.3 which is easily derived by an appropriate specialization in Proposition 5.2 and use of the relation $P(2, 1) = 2$.

Proposition 5.5. *For $k \geq 1$, $\ell_1, \dots, \ell_k \geq 1$ and $\mathbf{c}^{(k+1)} \neq \epsilon$, we have*

$$(5.6) \quad \begin{aligned} & P(\mathbf{c}^{(1)} \oplus (3^{\ell_1}) \oplus \mathbf{c}^{(2)} \oplus (3^{\ell_2}) \oplus \cdots \oplus \mathbf{c}^{(k)} \oplus (3^{\ell_k}) \oplus \mathbf{c}^{(k+1)}) \\ &= \left(\frac{1}{3}\right)^{\sum_{i=1}^k \ell_i - k} \frac{(3(\ell_1 + \cdots + \ell_k) + \sum_{i=1}^{k+1} |c_i|)!}{(1 + |\mathbf{c}^{(1)}|)! (\prod_{i=2}^k (3 + |\mathbf{c}^{(i)}|)!) \cdots (|\mathbf{c}^{(k+1)}| + 2)!} \\ & \quad \times P(\mathbf{a} \oplus (1)) \left(\prod_{i=1}^k P((2) \oplus \mathbf{c}^{(i)} \oplus (1)) \right) P((2) \oplus \mathbf{c}^{(k+1)}). \end{aligned}$$

Corollary 5.6. *For $\ell \geq 2$, $s, m \geq 1$, $t \geq 0$, we have*

$$(5.7) \quad P(3^\ell) = P(4, 3^{\ell-1}, 2) = \frac{1}{5} 3^{2-\ell} (3\ell)!,$$

$$(5.8) \quad P(3^\ell, 2) = 3^{-\ell} (3\ell + 2)!,$$

$$(5.9) \quad P(3^s, 2, 3^m) = P(4, 3^{m-1}, 2, 3^{s-1}, 2) = \frac{2}{25} 3^{-s-m} (3s + 3m + 2)!,$$

$$(5.10) \quad P(3^s, 2, 3^t, 2) = P(3^{t+1}, 2, 3^{s-1}, 2) = \frac{2}{5} 3^{-s-t} (3s + 3t + 4)!,$$

$$(5.11) \quad P(4, 3^\ell) = \frac{1}{25} 3^{2-\ell} (3\ell + 4)!.$$

Proof. Specializing (5.6) at $k = 1$, $\mathbf{c}^{(1)} = \epsilon$, $\mathbf{c}^{(2)} = (3)$ and $\ell_1 = \ell - 1$ gives

$$P(3^\ell) = \left(\frac{1}{3}\right)^{\ell-2} \frac{(3\ell)!}{1!5!} P(\epsilon \oplus (1)) P((2) \oplus (3)),$$

which simplifies to (5.7) by using (5.3). Similarly, specializing (5.6) at $k = 2$, $\mathbf{c}^{(1)} = \epsilon$, $\mathbf{c}^{(2)} = \mathbf{c}^{(3)} = (2)$, $\ell_1 = s$ and $\ell_2 = t$ gives

$$P(3^s, 2, 3^t, 2) = \left(\frac{1}{3}\right)^{s+t-2} \frac{(3s+3t+4)!}{1!5!4!} P(\epsilon \oplus (1)) P((2) \oplus (2) \oplus (1)) P((2) \oplus (3)),$$

which simplifies to (5.10) by using (5.3). The proof of the other assertions are left to the reader. \square

6. PROOF OF THEOREM 1.1

The purpose of this section is to complete the proof of Theorem 1.1. We shall use the following result which is a direct consequence of Proposition 5.2.

Proposition 6.1. *Let $(\mathbf{c}^{(i)})_{0 \leq i \leq k}$ be a sequence of compositions such that $\mathbf{c}^{(k)} \neq \epsilon$. Then, for any permutation $\sigma \in \mathcal{S}_{k-1}$, we have*

$$(6.1) \quad P(\mathbf{c}^{(0)} \oplus (3) \oplus \mathbf{c}^{(1)} \oplus (3) \oplus \cdots \oplus \mathbf{c}^{(k)}) = P(\mathbf{c}^{(0)} \oplus (3) \oplus \mathbf{c}^{(\sigma_1)} \oplus (3) \oplus \cdots \oplus \mathbf{c}^{(\sigma_{k-1})} \oplus (3) \oplus \mathbf{c}^{(k)}).$$

Combining Corollary 5.4 with the above result leads to the following result.

Corollary 6.2. *Let $\mathbf{c} = (c_1, \dots, c_k)$ be a maximal composition with $k \geq 3$.*

- (1) *If $c_1 = 3$, then \mathbf{c} has no part equal to 4.*
- (2) *If $c_1 = 4$, then \mathbf{c} has only one part equal to 4.*
- (3) *If $c_1 = c_k = 3$, then \mathbf{c} has at most one part equal to 2.*
- (4) *If $c_1 = 3$ and $c_k = 2$, then \mathbf{c} has at most two part equals to 2.*
- (5) *If $c_1 = 4$, and \mathbf{c} has a part equal to 2 then $\mathbf{c} = (4, 3^s, 2^t)$ for some $s, t \geq 1$, $t \leq 2$.*

Proof. By Corollary 5.4, we can suppose that $\mathbf{c} = x_0 \oplus (3) \oplus x_1 \oplus \cdots \oplus (3) \oplus x_s$ for some $s \geq 1$ and some sequence of compositions $(x_i)_i$ such that $x_0 \in \{\epsilon, (4)\}$, $x_s \in \{\epsilon, (2), (2, 2)\}$ and $x_i \in \{\epsilon, (2), (4)\}$ for $i = 1, \dots, s-1$.

(1) Suppose $c_1 = 3$ and $c_i = 4$ for some $i \geq 2$ (i.e., $x_0 = \epsilon$ and $x_j = (4)$ for some j with $1 \leq j < s$). By (6.1), we can assume that $x_1 = (4)$ (i.e., $c_2 = 4$). In this case, $r'\mathbf{c} = (4, 2) \oplus \mathbf{u}$ for $\mathbf{u} \neq \epsilon$ and thus, \mathbf{c} is not maximal by Lemma 4.2. This contradicts our assumption. Thus $c_i \neq 4$ for $2 \leq i \leq k$.

(2) Suppose $c_1 = 4$ and $c_i = 4$ for some $i \geq 2$ (i.e., $x_0 = x_j = (4)$ for some j with $1 \leq j < s$). By (6.1), we can assume that $x_1 = (4)$ (i.e., $c_3 = 4$). In this case, $\mathbf{c} = (4, 3, 4) \oplus \mathbf{b}$ for some $\mathbf{b} \neq \epsilon$, and thus, \mathbf{c} is not maximal by Lemma 4.3.

(3) Suppose $c_1 = c_k = 3$ and \mathbf{c} has at least two parts equal to 2. By (6.1), we can assume that $x_1 = x_2 = (2)$. In this case, $\mathbf{c} = (3, 2, 3, 2) \oplus \mathbf{b}$ for some $\mathbf{b} \neq \epsilon$, and thus, \mathbf{c} is not maximal by Lemma 4.3.

(4) Suppose $c_1 = 3$, $c_k = 2$ and \mathbf{c} has at least three parts equal to 2. There are two cases: $x_s = (2, 2)$ and $x_s = (2)$. If $x_s = (2, 2)$ (resp., $x_s = (2)$), by (6.1), we can assume that $x_{s-1} = (2)$ (resp., $x_{s-2} = x_{s-1} = (2)$). In this case, $\mathbf{c} = \mathbf{a} \oplus (2, 3, 2, 2)$ (resp., $\mathbf{c} = \mathbf{a} \oplus (2, 3, 2, 3, 2)$) for some $\mathbf{a} \neq \epsilon$. In both cases, \mathbf{c} is not maximal by Lemma 4.3.

(5) Suppose $c_1 = 4$ and $x_j = 2$ for some j with $1 \leq j < s$. By (6.1), we can assume that $x_1 = (2)$. In this case, $\mathbf{c} = (4, 3, 2) \oplus \mathbf{b}$ for some $\mathbf{b} \neq \epsilon$, and thus, \mathbf{c} is not maximal by Lemma 4.3. \square

Combining Corollaries 5.4 and 6.2 shows that

(6.2) *if \mathbf{c} is a maximal composition with at least three parts, then*

$$\mathbf{c} \in \{(3^\ell), (4, 3^\ell), (4, 3^{\ell-2}, 2), (4, 3^{\ell-2}, 2, 2), (3^\ell, 2), (3^\ell, 2, 2), (3^s, 2, 3^t, 2) : \ell \geq 2, s \geq 1, t \geq 0\},$$

which is very close to Theorem 1.1. On the other hand, using Fact 2.2 and Corollary 5.6, we see that for $s \geq 1, t \geq 0, k \geq 1$ and $0 \leq j \leq k-1, \ell \geq 2$, we have

$$(6.3) \quad \begin{aligned} P(4, 3^{\ell-2}, 2) &= P(3^\ell), & P(4, 3^s) &< P(3^s, 2, 2), & P(3^s, 2, 3^t) &< P(3^{s+t}, 2), \\ P(4, 3^{\ell-2}, 2, 2) &< P(3^\ell, 2), & P(3^{k-j}, 2, 3^j, 2) &= P(3^k, 2, 2). \end{aligned}$$

Combining (6.2) with (6.3) immediately implies that Theorem 1.1 is true for compositions with at least three parts. The validity of Theorem 1.1 for compositions with two parts (it suffices, by Proposition 3.1, to consider compositions with parts less than 5) can be treated by computer or by hand, and so, it is left to the reader.

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